Constraint equations for vacuum Einstein equations with a $\mathbb{S}^1$ symmetry

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Abstract

We solve the constraint equations for a vacuum space-time with a $\mathbb{S}^1$ symmetry satisfying the vacuum Einstein equations. Vacuum Einstein equations with a $\mathbb{S}^1$ symmetry have been studied by Choquet-Bruhat and Moncrief in the compact case, and by Ashtekar, Bicak and Schmidt in the case where an additional spherical symmetry is added. In this paper we consider the asymptotically flat case. This corresponds to solving a nonlinear elliptic system on $\mathbb{R}^2$. The main difficulty in that case is due to the delicate inversion of the Laplacian on $\mathbb{R}^2$.

1 Introduction

Einstein equations can be formulated as a Cauchy problem whose initial data must satisfy compatibility conditions known as the constraint equations. In this paper, we will consider the constraint equations for the vacuum Einstein equations, in the particular case where the space-time possesses a $\mathbb{S}^1$ symmetry. It allows for a reduction of the $3+1$ dimensional problem to a $2+1$ dimensional one. This symmetry has been studied by Yvonne Choquet-Bruhat and Vincent Moncrief in [3] (see also [4]) in the case of a space-time of the form $\Sigma \times \mathbb{S}^1 \times \mathbb{R}$, where Σ is a compact two dimensional manifold of genus $G \geq 2$, and $\mathbb{R}$ is the time axis, with a space-time metric independent of the $\mathbb{S}^1$ coordinate. They prove the existence of global solutions corresponding to perturbation of particular expanding initial data.

In this paper we consider a space-time of the form $\mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{R}$. It is an interesting problem to ask whether the global existence of solutions corresponding to small perturbations of the trivial initial data also holds in this case. However, it is crucial, before considering this problem, to ensure the existence of compatible initial data, i.e. the existence of solutions to the constraint equations. This is the purpose of the present paper.

In the compact case, the issue of solving the constraint equations is straightforward. Every metric on a compact manifold of genus $G \geq 2$ is conformal to a metric of scalar curvature -1. As a consequence, it is possible to decouple the system into elliptic scalar equations of the form $\Delta u = f(x,u)$ with $\partial_u f > 0$, for which existence results are standard (see for example chapter 14 in [9]).

The asymptotically flat case is more challenging. First, the definition of an asymptotically flat manifold is not so clear in two dimension. Ashtekar, Bicak and Schmidt constructed in [1] radial solutions of the $2+1$ dimensional problem with an angle at space-like infinity. In particular, these solutions do not tend to the Euclidean metric at space-like infinity. Moreover, the behaviour of the Laplace operator on $\mathbb{R}^2$ makes the issue of finding solutions to the constraint equations more intricate.

1.1 Reduction of the Einstein equations

Before discussing the constraint equations, we first briefly recall the form of the Einstein equations in the presence of a $\mathbb{S}^1$ symmetry. We follow here the exposition in [4]. A $\mathbb{S}^1$ symmetric metric ${}^{(4)}\mathbf{g}$ on $\mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{R}$ has the form

$$ {}^{(4)}\mathbf{g} = \widetilde{\mathbf{g}} + e^{2\gamma}(dx^3 + A_\alpha dx^\alpha)^2, $$

where $\widetilde{\mathbf{g}}$ is a Lorentzian metric on $\mathbb{R}^{2+1}$, γ is a scalar function on $\mathbb{R}^{1+2}$, A is a 1-form on $\mathbb{R}^{1+2}$, x^3 is the coordinate on $\mathbb{S}^1$ and x^α, $\alpha = 0,1,2$, are the coordinates on $\mathbb{R}^{2+1}$. Since $\mathbb{S}^1$ is an isometry group, $\mathbf{g}$, γ and A do not depend on x^3. We set $F = dA$, where d is the exterior differential. F is then a 2-form. Let also ${}^{(4)}\mathbf{R}_{\mu\nu}$ denote the Ricci tensor associated to ${}^{(4)}\mathbf{g}$. $\widetilde{\mathbf{R}}_{\alpha\beta}$ and $\widetilde{\mathbf{D}}$ are respectively the Ricci tensor and the covariant derivative associated to $\widetilde{\mathbf{g}}$.

With this metric, the vacuum Einstein equations

$$ {}^{(4)}\mathbf{R}_{\mu\nu} = 0, \ \mu,\nu = 0,1,2,3 $$

can be written in the basis $(dx^\alpha, dx^3 + A_\alpha dx^\alpha)$ (see [4] appendix VII)

$$ 0 = {}^{(4)}\mathbf{R}_{\alpha\beta} = \widetilde{\mathbf{R}}_{\alpha\beta} - \frac{1}{2}e^{2\gamma}F_\alpha{}^\lambda F_{\beta\lambda} - \widetilde{\mathbf{D}}_\alpha\partial_\beta\gamma - \partial_\alpha\gamma\partial_\beta\gamma, \tag{1} $$

$$ 0 = {}^{(4)}\mathbf{R}_{\alpha 3} = \frac{1}{2}e^{-\gamma}\widetilde{\mathbf{D}}_\beta(e^{3\gamma}F_\alpha{}^\beta), \tag{2} $$

$$ 0 = {}^{(4)}\mathbf{R}_{33} = -e^{-2\gamma}\left(-\frac{1}{4}e^{2\gamma}F_{\alpha\beta}F^{\alpha\beta} + \widetilde{\mathbf{g}}^{\alpha\beta}\partial_\alpha\gamma\partial_\beta\gamma + \widetilde{\mathbf{g}}^{\alpha\beta}\widetilde{\mathbf{D}}_\alpha\partial_\beta\gamma\right). \tag{3} $$

The equation (2) is equivalent to

$$ d(*e^{3\gamma}F) = 0 $$

where $*e^{3\gamma}F$ is the adjoint one form associated to $e^{3\gamma}F$. This is equivalent, on $\mathbb{R}^{1+2}$, to the existence of a potential ω such that

$$ *e^{3\gamma}F = d\omega. $$

Since F is a closed 2-form, we have $dF = 0$. By doing the conformal change of metric $\widetilde{\mathbf{g}} = e^{-2\gamma}\mathbf{g}$, this equation, together with the equations (1) and (3), yield the following system,

$$ \Box_{\mathbf{g}}\omega - 4\partial^\alpha\gamma\partial_\alpha\omega = 0, \tag{4} $$

$$ \Box_{\mathbf{g}}\gamma + \frac{1}{2}e^{-4\gamma}\partial^\alpha\omega\partial_\alpha\omega = 0, \tag{5} $$

$$ \mathbf{R}_{\alpha\beta} = 2\partial_\alpha\gamma\partial_\beta\gamma + \frac{1}{2}e^{-4\gamma}\partial_\alpha\omega\partial_\beta\omega, \ \alpha,\beta = 0,1,2, \tag{6} $$

where $\Box_{\mathbf{g}}$ is the d'Alembertian[1] in the metric $\mathbf{g}$ and $\mathbf{R}_{\alpha\beta}$ is the Ricci tensor associated to $\mathbf{g}$. We introduce the following notation

$$ \partial_\alpha u\partial_\beta u = 2\partial_\alpha\gamma\partial_\beta\gamma + \frac{1}{2}e^{-4\gamma}\partial_\alpha\omega\partial_\beta\omega. \tag{7} $$

We consider the Cauchy problem for the equations (4), (5) and (6). As it is in the case for the $3+1$ Einstein equation, the initial data for (4), (5) and (6) cannot be prescribed arbitrarily. They have to satisfy constraint equations.

[1] $\Box_{\mathbf{g}}$ is the Lorentzian equivalent of the Laplace-Beltrami operator in Riemannian geometry. In a coordinate system, we have $\Box_{\mathbf{g}}u = \frac{1}{\sqrt{|\mathbf{g}|}}\partial_\alpha(\mathbf{g}^{\alpha\beta}\sqrt{|\mathbf{g}|}\partial_\beta u)$.

1.2 Constraint equations

We can write the metric $\mathbf{g}$ under the form

$$\mathbf{g} = -N^2(dt)^2 + g_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt), \tag{8}$$

where the scalar function N is called the lapse, the vector field β is called the shift and g is a Riemannian metric on $\mathbb{R}^2$.

We consider the initial space-like surface $\mathbb{R}^2 = \{t = 0\}$. We will use the notation

$$\partial_0 = \partial_t - \mathcal{L}_\beta,$$

where $\mathcal{L}_\beta$ is the Lie derivative associated to the vector field β. With this notation, we have the following expression for the second fundamental form of $\mathbb{R}^2$

$$K_{ij} = -\frac{1}{2N}\partial_0 g_{ij}.$$

We will use the notation

$$\tau = g^{ij}K_{ij}$$

for the mean curvature. We also introduce the Einstein tensor

$$\mathbf{G}_{\alpha\beta} = \mathbf{R}_{\alpha\beta} - \frac{1}{2}\mathbf{R}\mathbf{g}_{\alpha\beta},$$

where $\mathbf{R}$ is the scalar curvature $\mathbf{R} = \mathbf{g}^{\alpha\beta}\mathbf{R}_{\alpha\beta}$. The constraint equations are given by

$$\mathbf{G}_{0i} \equiv N(\partial_j\tau - D^i K_{ij}) = \partial_0 u \partial_j u, \ i = 1, 2, \tag{9}$$

$$\mathbf{G}_{00} \equiv \frac{N^2}{2}(R - |K|^2 + \tau^2) = \partial_0 u \partial_0 u - \frac{1}{2}\mathbf{g}_{00}\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u, \tag{10}$$

where u has been defined in (7) and D and R are respectively the covariant derivative and the scalar curvature associated to g (see [4] chapter VI for a derivation of (9) and (10)). (9) is called the momentum constraint and (10) is called the Hamiltonian constraint.

We will look for g of the form $g = e^{2\lambda}\delta$ where δ is the Euclidean metric on $\mathbb{R}^2$. There is no loss of generality since, up to a diffeomorphism, all metrics on $\mathbb{R}^2$ are conformal to the Euclidean metric. We introduce the traceless part of K,

$$H_{ij} = K_{ij} - \frac{1}{2}\tau g_{ij},$$

and following [3] we introduce the quantity

$$\dot{u} = \frac{e^{2\lambda}}{N}\partial_0 u.$$

Then the equations (9) and (10) take the form

$$\partial_i H_{ij} = -\dot{u}\partial_j u + \frac{1}{2}e^{2\lambda}\partial_j \tau, \tag{11}$$

$$\Delta\lambda + e^{-2\lambda}\left(\frac{1}{2}\dot{u}^2 + \frac{1}{2}|H|^2\right) - e^{2\lambda}\frac{\tau^2}{4} + \frac{1}{2}|\nabla u|^2 = 0. \tag{12}$$

The aim of this paper is to solve the coupled system of nonlinear elliptic equations (11) and (12) on $\mathbb{R}^2$ in the small data case, that is to say when $\dot{u}$ and ∇u are small. A similar system can be obtained when studying the constraint equations in three dimensions. The strategy in the asymptotically flat three-dimensional case is to set $\tau = 0$. Then the constraint equations decouple and the difficulty that remains is the study of the scalar equation (12), also called the Lichnerowicz equation[2]. The asymptotically flat three-

[2]The resolution of this equation is closely linked to the Yamabe problem

dimensional case has been studied first in [2] (see chapter 7 in [4] and references therein for more details on the resolution of the constraint equations in this case). In the compact case there have been also some results concerning the coupled constraint equations, i.e. without setting τ constant (see [6]).

In our case, the difficulty will arise from particular issues concerning the inversion of second order elliptic operators on $\mathbb{R}^2$. In particular, it is not possible to set $\tau = 0$ in the case of $\mathbb{R}^2$. Indeed, equation (11) induces for H the asymptotic $|H|^2 \sim \frac{1}{r^2}$ as r tends to infinity. Now, it is known (see [8]) that an equation of the form

$$\Delta u + Re^{2u} + f = 0,$$

with $R, f \leq 0$ and $R \lesssim -\frac{1}{r^2}$ when r tends to infinity, admits no solution. Therefore, we will be forced to carefully adjust the asymptotic behaviour of τ as r tends to infinity, to compensate the term $|H|^2$ in equation (12), and to ensure that we remain in the range of the elliptic operators which come into play.

Remark 1.1. *The solutions of equation* (12) *that we construct in this paper satisfy*

$$\lambda = -\alpha \ln(r) + o(1)$$

as $r \to \infty$. Using a change of variable, we observe that this asymptotic behaviour is equivalent to the presence of an asymptotic angle at space-like infinity. Indeed, if we make the change of coordinate $r' = r^{1-\alpha}$ for r large enough, then the metric

$$g \sim r^{-2\alpha}(dr^2 + r^2 d\theta^2), \quad r \to \infty$$

takes the form

$$g' \sim \frac{1}{(1-\alpha)^2} dr'^2 + r'^2 d\theta^2, \quad r' \to \infty$$

which corresponds to a conical singularity at space-like infinity, with an angle given by

$$2\pi(1-\alpha).$$

Note that, since the constraint equations (9) *and* (10) *are independent of the choice of coordinates, the metric g' and the second fondamental form K', obtained by performing the change of variables $r' = r^{1-\alpha}$ for r large enough, are still solutions of the constraint equations.*

We will do the following rescaling to avoid the $e^{2\lambda}$ and $e^{-2\lambda}$ factors

$$\breve{u} = e^{-\lambda}\dot{u}, \quad \breve{H} = e^{-\lambda}H, \quad \breve{\tau} = e^{\lambda}\tau.$$

Then the equations (11) and (12) become

$$\partial_i \breve{H}_{ij} + \breve{H}_{ij}\partial_i\lambda = -\breve{u}\partial_j u + \frac{1}{2}\partial_j\breve{\tau} - \frac{1}{2}\breve{\tau}\partial_j\lambda,$$
$$\Delta\lambda + \frac{1}{2}\breve{u}^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\breve{H}|^2 - \frac{\breve{\tau}^2}{4} = 0.$$

To lighten the notations, we will omit the $\breve{}$ in the rest of the paper.

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2 Main result

We are interested in the system of constraint equations on $\mathbb{R}^2$

$$\begin{cases} \partial_i H_{ij} + H_{ij}\partial_i \lambda = -\dot{u}\partial_j u + \frac{1}{2}\partial_j \tau - \frac{1}{2}\tau \partial_j \lambda, \\ \Delta\lambda + \frac{1}{2}\dot{u}^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|H|^2 - \frac{\tau^2}{4} = 0. \end{cases} \tag{13}$$

We look for solutions (H, λ) where H is a 2-tensor, symmetric and traceless, and λ is a scalar function. $\dot{u}$ and u are given scalar functions on $\mathbb{R}^2$ and τ is a scalar function which may be chosen arbitrarily. We recall however from the end of the previous section that τ must be chosen carefully. In particular, τ can not be identically 0 otherwise there may not be solutions.

Before stating the main theorem, we recall several properties of weighted Sobolev spaces.

2.1 Weighted Sobolev spaces

In the rest of the paper, $\chi(r)$ denotes a smooth non negative function such that

$$0 \leq \chi \leq 1, \quad \chi(r) = 0 \text{ for } r \leq 1, \quad \chi(r) = 1 \text{ for } r \geq 2.$$

We will also note $f \lesssim h$ when there exists a universal constant C such that $f \leq Ch$.

Definition 2.1. *Let $m \in \mathbb{N}$ and $\delta \in \mathbb{R}$. The weighted Sobolev space $H^m_\delta(\mathbb{R}^n)$ is the completion of C^∞_0 for the norm*

$$\|u\|_{H^m_\delta} = \sum_{|\beta|\leq m} \|(1+|x|^2)^{\frac{\delta+|\beta|}{2}} D^\beta u\|_{L^2}.$$

The weighted Hölder space C^m_δ is the completion of C^∞_0 for the norm

$$\|u\|_{C^m_\delta} = \sum_{|\beta|\leq m} \|(1+|x|^2)^{\frac{\delta+|\beta|}{2}} D^\beta u\|_{L^\infty}.$$

Let $0 < \alpha < 1$. The Hölder space $C^{m+\alpha}_\delta$ is the completion of C^∞_0 for the norm

$$\|u\|_{C^{m+\alpha}_\delta} = \|u\|_{C^m_\delta} + \sup_{x\neq y} \frac{|\partial^m u(x) - \partial^m u(y)|(1+|x|^2)^{\frac{\delta}{2}}}{|x-y|^\alpha}.$$

The following lemma is an immediate consequence of the definition.

Lemma 2.2. *Let $m \in \mathbb{N}$ and $\delta \in \mathbb{R}$. Then $u \in H^m_\delta$ implies $\partial_j u \in H^{m-1}_{\delta+1}$ for $j = 1, .., n$.*

We first recall the Sobolev embedding with weights (see for example [4], Appendix I). In the rest of this section, we assume $n = 2$.

Proposition 2.3. *Let $s, m \in \mathbb{N}$. We assume $s > 1$. Let $\beta \leq \delta + 1$ and $0 < \alpha < min(1, s-1)$. Then, we have the continuous embedding*

$$H^{s+m}_\delta \subset C^{m+\alpha}_\beta.$$

We will also need a product rule.

Proposition 2.4. *Let $s, s_1, s_2 \in \mathbb{N}$. We assume $s \leq \min(s_1, s_2)$ and $s < s_1 + s_2 - 1$. Let $\delta < \delta_1 + \delta_2 + 1$. Then $\forall (u, v) \in H^{s_1}_{\delta_1} \times H^{s_2}_{\delta_2}$,*

$$\|uv\|_{H^s_\delta} \lesssim \|u\|_{H^{s_1}_{\delta_1}} \|v\|_{H^{s_2}_{\delta_2}}.$$

The following simple lemma will be useful as well.

Lemma 2.5. *Let $\alpha \in \mathbb{R}$ and $g \in L^\infty$ be such that*

$$|g(x)| \lesssim (1 + |x|^2)^\alpha.$$

Then the multiplication by g maps H^0_δ to $H^0_{\delta - 2\alpha}$.

We have the following theorem due to Mac Owen (see [7])

Theorem 2.6. *(Theorem 0 in [7]) Let $m \in \mathbb{N}$ and $-1 + m < \delta < m$. The Laplace operator $\Delta : H^2_\delta \to H^0_{\delta+2}$ is an injection with closed range*

$$\left\{ f \in H^0_{\delta+2} \mid \int fv = 0 \quad \forall v \in \cup_{i=0}^m \mathcal{H}_i \right\},$$

where $\mathcal{H}_i$ is the set of harmonic polynomials of degree i. Moreover, u obeys the estimate

$$\|u\|_{H^2_\delta} \leq C(\delta) \|\Delta u\|_{H^0_{\delta+2}},$$

where $C(\delta)$ is a constant such that $C(\delta) \to +\infty$ when $\delta \to m_-$ and $\delta \to (-1+m)_+$.

Corollary 2.7. *Let $-1 < \delta < 0$ and $f \in H^0_{\delta+2}$. Then there exists a solution u of*

$$\Delta u = f$$

which can be written

$$u = \left(\int f \right) \chi(r) ln(r) + v,$$

where $v \in H^2_\delta$ is such that $\|v\|_{H^2_\delta} \leq C(\delta) \|f\|_{H^0_{\delta+2}}$.

Proof. Let Θ be a smooth compactly supported function such that $\int \Theta = 1$. Let u_0 be a solution of $\Delta u_0 = \Theta$. The representation formula, based on the fundamental solution of the Laplacian on $\mathbb{R}^2$ yields

$$u_0(x) = \int ln(|x - y|) \Theta(y) dy.$$

Since Θ is compactly supported we have that

$$u_0(x) = \chi(|x|) ln(|x|) + \widetilde{u_0},$$

with $\widetilde{u_0} \in H^2_\delta$. Theorem 2.6 implies that there exists $\widetilde{v} \in H^2_\delta$ solution of

$$\Delta \widetilde{v} = f - \left(\int f \right) \Theta.$$

Therefore

$$u = \left(\int f \right) \chi(r) ln(r) + \left(\int f \right) \widetilde{u_0} + \widetilde{v}$$

is a solution of $\Delta u = f$. To obtain the estimate of the corollary, it suffices to note that, for $f \in H^0_{\delta+2}$ we have

$$\int |f| = \int |f| \frac{(1 + r^2)^{\frac{\delta}{2}+1}}{(1 + r^2)^{\frac{\delta}{2}+1}} \lesssim \frac{1}{\sqrt{1 + \delta}} \|f\|_{H^0_{\delta+2}}.$$

$\square$

Corollary 2.8. *Let $s, m \in \mathbb{N}$ and $-1+m < \delta < m$. The Laplace operator $\Delta : H^{2+s}_\delta \to H^s_{\delta+2}$ is an injection with closed range*

$$\left\{ f \in H^s_{\delta+2} \mid \int fv = 0 \quad \forall v \in \cup_{i=0}^m \mathcal{H}_i \right\}.$$

Moreover, u obeys the estimate

$$\|u\|_{H^{s+2}_\delta} \leq C(s,\delta)\|\Delta u\|_{H^s_{\delta+2}}.$$

Proof. We will proceed by induction. Note that Theorem 2.6 corresponds to the case $s = 0$. We assume that the statement of the corollary holds true for some $s \in \mathbb{N}$ and all $m \in \mathbb{N}$, and we will prove that it holds true for $s+1$. Let $m \in \mathbb{N}$ and $-1+m < \delta < m$. Let $f \in H^{s+1}_{\delta+2}$, such that f belongs to the set

$$\left\{ f \in H^0_{\delta+2} \mid \int fv = 0 \quad \forall v \in \cup_{i=0}^m \mathcal{H}_i \right\}.$$

Then Theorem 2.6 provides a unique $u \in H^2_\delta$ such that $\Delta u = f$. In particular for $i = 1, 2$ we have

$$\Delta \partial_i u = \partial_i f.$$

Since $f \in H^{s+1}_{\delta+2}$, we have that $\partial_i f \in H^s_{\delta+3}$. Moreover, for all v, harmonic polynomial of degree $j \leq m+1$, we have

$$\int (\partial_i f) v = -\int f \partial_i v = 0,$$

because $\partial_i v$ is an harmonic polynomial of degree $j - 1 \leq m$. Therefore, by induction, we have $\partial_i u \in H^{s+2}_{\delta+1}$ and

$$\begin{aligned}
\|u\|_{H^{s+1+2}_\delta} &\lesssim \|u\|_{H^2_\delta} + \|\partial_1 u\|_{H^{s+2}_{\delta+1}} + \|\partial_2 u\|_{H^{s+2}_{\delta+1}} \\
&\leq C(\delta)\|f\|_{H^0_{\delta+2}} + C(s, \delta+1)\left(\|\partial_1 f\|_{H^s_{\delta+3}} + \|\partial_2 f\|_{H^s_{\delta+3}}\right) \\
&\leq C(s+1, \delta)\|f\|_{H^{s+1}_{\delta+2}}.
\end{aligned}$$

$\square$

2.2 Main result

In the rest of the paper, δ will be a fixed real number such that

$$-1 < \delta < 0.$$

Definition 2.9. *Let $\delta' \in \mathbb{R}$ and $s \in \mathbb{N}$. We note $\mathcal{H}^s_{\delta'}$ the set of symmetric traceless tensor whose components are in $H^s_{\delta'}$,*

The following theorem is our main result.

Theorem 2.10. *Let $\dot{u}^2, |\nabla u|^2 \in H^0_{\delta+2}$, $\widetilde{\tau} \in H^1_{\delta+1}$ and $b \in \mathbb{R}$. We note*

$$\varepsilon = \int \dot{u}^2 + |\nabla u|^2.$$

We assume

$$\|\dot{u}^2\|_{H^0_{\delta+2}} + \||\nabla u|^2\|_{H^0_{\delta+2}} + \|\widetilde{\tau}\|_{H^1_{\delta+1}} + |b| \lesssim \varepsilon.$$

If $\varepsilon > 0$ is small enough, there exist $\alpha, \rho, \eta \in \mathbb{R}$, a scalar function $\widetilde{\lambda} \in H^2_\delta$ and a traceless symmetric tensor $\widetilde{H} \in \mathcal{H}^1_{\delta+1}$ such that, if we note

$$\begin{aligned} H = & -\frac{b\chi(r)}{r}\begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix} - \frac{\rho\chi(r)}{4r}\begin{pmatrix} \cos(\theta+\eta) & \sin(\theta+\eta) \\ \sin(\theta+\eta) & -\cos(\theta+\eta) \end{pmatrix} \\ & - \frac{\rho\chi(r)}{4r}\begin{pmatrix} \cos(3\theta-\eta) & \sin(3\theta-\eta) \\ \sin(3\theta-\eta) & -\cos(3\theta-\eta) \end{pmatrix} + \widetilde{H}, \\ \lambda = & -\alpha\chi(r)\ln(r) + \widetilde{\lambda}, \end{aligned}$$

then λ, H is a solution of (13) *with*

$$\tau = \frac{b\chi(r)}{r} + \frac{\rho\chi(r)}{r}\cos(\theta - \eta) + \widetilde{\tau}.$$

Moreover, $\alpha, \rho, \eta, \widetilde{\lambda}, \widetilde{H}$ are unique. Finally, α, ρ, η are such that

$$\begin{aligned} \alpha &= \frac{1}{2}\int \left(\dot{u}^2 + |\nabla u|^2\right) + O(\varepsilon^2), \\ \rho\cos(\eta) &= \frac{4}{1+2\pi}\int \dot{u}\partial_1 u + O(\varepsilon^2), \\ \rho\sin(\eta) &= \frac{4}{1+2\pi}\int \dot{u}\partial_2 u + O(\varepsilon^2), \end{aligned}$$

and we have the estimates $\|\widetilde{\lambda}\|_{H^2_\delta} \lesssim \varepsilon$ and $\|\widetilde{H}\|_{\mathcal{H}^1_{\delta+1}} \lesssim \varepsilon$.

The following corollary is an immediate consequence of Theorem 2.10 and Corollary 2.8.

Corollary 2.11. *Let $b \in \mathbb{R}$. Let $s \in \mathbb{N}$ and assume $\dot{u}^2, |\nabla u|^2 \in H^s_{\delta+2}$ and $\widetilde{\tau} \in H^{s+1}_{\delta+1}$. Let ε be defined as in Theorem 2.10. Then the conclusion of Theorem 2.10 holds and we have furthermore $\widetilde{\lambda} \in H^{s+2}_\delta$, $\widetilde{H} \in \mathcal{H}^{s+1}_{\delta+1}$, with the estimates*

$$\|\widetilde{\lambda}\|_{H^{s+2}_\delta} + \|\widetilde{H}\|_{\mathcal{H}^{s+1}_{\delta+1}} \lesssim \|\dot{u}^2\|_{H^s_{\delta+2}} + \||\nabla u|^2\|_{H^s_{\delta+2}} + \|\widetilde{\tau}\|_{H^{s+1}_{\delta+1}} + |b|.$$

Comments on Theorem 2.10

1. The trivial asymptotically flat solution to the Einstein vacuum equations with $\mathbb{S}^1$ symmetry (4), (5) and (6) is obtained by taking for $\mathbf{g}$ the Minkowski metric on $\mathbb{R}^{1+2}$, and by setting $\omega = \gamma = 0$. The corresponding initial data set is given by

   $$(\dot{u} = 0, u = 0, \tau = 0, H = 0, \lambda = 0).$$

   Theorem 2.10 corresponds to the existence of solutions to the constraint equations which are small perturbations of $(0,0,0,0,0)$. An interesting open problem is the question of the non linear stability of the "Minkowski space-time with $\mathbb{S}^1$ symmetry" under these perturbations[3].

2. We solve here the constraint equations for small data. It is an interesting open problem to investigate the large data case.

[3]This is the analogue in dimension $2+1$ of the nonlinear stability of the Minkowski space-time in dimension $3+1$, which has been established in the celebrated work of Christodoulou and Klainerman [5].

3. We look for asymptotically flat solutions. However, we mentioned in the introduction that it is not so clear what to expect as a definition of asymptotic flatness in $2+1$ dimension. The solutions of the evolution problem (4), (5) and (6) with an additional spherical symmetry and $\omega \equiv 0$, known as Einstein-Rosen waves, have been studied in [1]. These solutions exhibit a conical singularity at space-like infinity, that is to say the perimeter of a circle of radius r asymptotically grows like $2\pi c r$ with $c < 1$, instead of $2\pi r$ in the Euclidean metric. The Riemannian metric we obtain in our case is asymptotically

$$g \sim r^{-2\alpha}(dr^2 + r^2 d\theta^2), \quad r \to \infty$$

If we perform the change of variable $r' = r^{1-\alpha}$ for r large enough, we obtain the new metric

$$g' \sim \frac{1}{(1-\alpha)^2} dr'^2 + r'^2 d\theta^2, \quad r' \to \infty$$

which is a metric with conical singularity at infinity, corresponding to an angle given by $2\pi(1-\alpha)$.

4. The role of the parameters ρ and η is to compensate the asymptotic behaviour $H = O(\frac{1}{r})$ which is created by the first equation in (13) and induces problems in the second equation. Indeed, the decay $H = O(\frac{1}{r})$ is not enough to ensure that $|H|^2 \in H^0_{\delta+2}$ when $-1 < \delta < 0$. Thus, the role of ρ and η is to induce a cancellation which allows the term $\frac{1}{2}|H|^2 - \frac{1}{4}\tau^2$ to belong to $H^0_{\delta+2}$. Indeed, it is necessary that the right-hand side of the second equation of (13) belongs to $H^0_{\delta+2}$ in order to solve this equation with Corollary 2.7.

5. The parameter b is a free parameter. It permits to have a space-time metric with a non trivial asymptotic shift β. This flexibility may be useful in the study of the evolution problem.

6. The quantities b, ρ and η are conserved by the flow of the Einstein equations. To see this, we note that these quantities can be expressed as

$$\begin{aligned} b &= \frac{1}{2\pi} \lim_{r\to\infty} \int_0^{2\pi} \tau r d\theta, \\ \rho\cos(\eta) &= \frac{1}{\pi} \lim_{r\to\infty} \int_0^{2\pi} \tau \cos(\theta) r d\theta, \\ \rho\sin(\eta) &= \frac{1}{\pi} \lim_{r\to\infty} \int_0^{2\pi} \tau \sin(\theta) r d\theta. \end{aligned}$$

The $(0,0)$ component of equation (6) can be written under the form

$$\partial_0 \tau = -e^{-2\lambda}\Delta N + \left(e^{-4\lambda}(|H|^2 + \dot{u}^2) + \frac{\tau^2}{2} \right) N.$$

This yields $\partial_t \tau = O(\frac{1}{r^2})$ as r tends to ∞ and therefore

$$\partial_t b = \partial_t(\rho\cos(\eta)) = \partial_t(\rho\sin(\eta)) = 0.$$

2.3 Outline of the proof

We will prove Theorem 2.10 by a fixed point argument.

The construction of the map F. We will construct a map

$$F : \mathbb{R} \times H^2_\delta \times \mathcal{H}^1_{\delta+1} \to \mathbb{R} \times H^2_\delta \times \mathcal{H}^1_{\delta+1}$$
$$(\alpha, \widetilde{\lambda}, \widetilde{H}) \mapsto (\alpha', \widetilde{\lambda}', \widetilde{H}'),$$

such that (λ', H') given by

$$\lambda' = -\alpha' \ln(r)\chi(r) + \widetilde{\lambda}',$$
$$H' = H_b + H_{\rho,\eta} + \widetilde{H}',$$

are solutions of

$$\partial_i H'_{ij} + H_{ij}\partial_i \lambda = -\dot{u}\partial_j u + \frac{1}{2}\partial_j \tau - \frac{1}{2}\tau \partial_j \lambda, \tag{14}$$

$$\Delta\lambda' + \frac{1}{2}\dot{u}^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|H|^2 - \frac{\tau^2}{4} = 0, \tag{15}$$

with λ, H, τ defined by

$$\lambda = -\alpha \ln(r)\chi(r) + \widetilde{\lambda}, \tag{16}$$

$$H = H_b + H_{\rho,\eta} + \widetilde{H}, \tag{17}$$

$$\tau = \frac{b\chi(r)}{r} + \frac{\rho\chi(r)}{r}\cos(\theta - \eta) + \widetilde{\tau}, \tag{18}$$

where we note here, and in the rest of the paper

$$H_b = -\frac{b\chi(r)}{2r}\begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix},$$

$$H_{\rho,\eta} = -\frac{\rho\chi(r)}{4r}\begin{pmatrix} \cos(\theta+\eta) & \sin(\theta+\eta) \\ \sin(\theta+\eta) & -\cos(\theta+\eta) \end{pmatrix} - \frac{\rho\chi(r)}{4r}\begin{pmatrix} \cos(3\theta-\eta) & \sin(3\theta-\eta) \\ \sin(3\theta-\eta) & -\cos(3\theta-\eta) \end{pmatrix},$$

with ρ, η depending on $b, \alpha, \widetilde{\lambda}, \widetilde{H}, \dot{u}, \nabla u$. Then, proving that F has a fixed point easily follows from the estimates derived for α', λ' and H', which concludes the proof of Theorem 2.10. Thus the core of the analysis is to solve (14) and (15).

Solving (14)**.** For λ, H, τ of the form (16), (17) and (18), there always exists a solution of (14) of the form

$$H' = \frac{\chi(r)}{r}\begin{pmatrix} g_1(\theta) & g_2(\theta) \\ g_2(\theta) & -g_1(\theta) \end{pmatrix} + \widetilde{H}',$$

where g_1 and g_2 are two functions of the angle θ, and $\widetilde{H}'$ is a traceless symmetric tensor belonging to $\mathcal{H}^1_{\delta+1}$. For $\varepsilon > 0$ small enough, we will be able to choose ρ and η such that H' can be written under the form $H_b + H_{\rho,\eta} + \widetilde{H}'$.

Solving (15)**.** We remark that, according to Theorem 2.6, the equation (15) may not have solutions in H^2_δ. Also, because of the asymptotic behaviour of $|H|^2$ and τ^2, which are only decreasing like $\frac{1}{r^2}$ as $r \to \infty$, the right-hand side of the equation (15) may not be in the space $H^0_{\delta+2}$, in which case we may not be able to solve the equation thanks to Corollary 2.7. However, the particular form of H and τ allows the terms decreasing like $\frac{1}{r^2}$ to balance each other, and we are able to obtain a solution of (15) of the form $-\alpha' \ln(r)\chi(r) + \widetilde{\lambda}'$ with

$$\alpha' = \int \left(\frac{1}{2}\dot{u}^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|H|^2 - \frac{\tau^2}{4}\right).$$

The rest of the paper is as follows. In section 3, we explain how to solve the momentum constraint (14). In section 4, we explain how to choose the coefficients ρ, η and how to solve the Lichnerowicz equation (15). Finally the map F is constructed in section 5. It is shown to have a fixed point, which concludes the proof of Theorem 2.10.

3 The momentum constraint

The goal of this section is to solve equation (14).

Proposition 3.1. *We assume $\dot{u}\nabla u \in H^0_{\delta+2}$. Let $\alpha, b, \rho, \eta \in \mathbb{R}$, and let*

$$\begin{aligned}
\tau &= b\frac{\chi(r)}{r} + \rho\frac{\chi(r)}{r}\cos(\theta-\eta) + \widetilde{\tau},\\
\lambda &= -\alpha\chi(r)\ln(r) + \widetilde{\lambda},\\
H &= H_b + H_{\rho,\eta} + \widetilde{H},
\end{aligned}$$

with $\widetilde{\tau} \in H^1_{\delta+1}$, $\widetilde{\lambda} \in H^2_\delta$, $\widetilde{H} \in \mathcal{H}^1_{\delta+1}$. Then the equation

$$\partial_i H'_{ij} + H_{ij}\partial_i\lambda = -\dot{u}\partial_j u + \frac{1}{2}\partial_j\tau - \frac{1}{2}\tau\partial_j\lambda,$$

has a unique solution of the form

$$\begin{aligned}
H' =& \frac{m\chi(r)}{r}\begin{pmatrix} \cos(\theta+\phi) & \sin(\theta+\phi)\\ \sin(\theta+\phi) & -\cos(\theta+\phi)\end{pmatrix}\\
&- \frac{\rho\chi(r)}{4r}\begin{pmatrix} \cos(3\theta-\eta) & \sin(3\theta-\eta)\\ \sin(3\theta-\eta) & -\cos(3\theta-\eta)\end{pmatrix} + H_b + \widetilde{H}',
\end{aligned}$$

with $\widetilde{H}' \in \mathcal{H}^1_{\delta+1}$ and

$$\begin{aligned}
m\cos(\phi) = \int\Big(& -\dot{u}\partial_1 u - \frac{1}{2}\widetilde{\tau}\partial_1\lambda - \widetilde{H}_{i1}\partial_i\lambda - \partial_i\widetilde{\lambda}(H_b+H_{\rho,\eta})_{i1}\\
&- \frac{1}{2}\chi(r)\frac{b+\rho\cos(\theta-\eta)}{r}\partial_1\widetilde{\lambda}\Big) + \frac{\pi\rho}{2}\cos(\eta),
\end{aligned} \tag{19}$$

$$\begin{aligned}
m\sin(\phi) = \int\Big(& -\dot{u}\partial_2 u - \frac{1}{2}\widetilde{\tau}\partial_2\lambda - \widetilde{H}_{i2}\partial_i\lambda - \partial_i\widetilde{\lambda}(H_b+H_{\rho,\eta})_{i2}\\
&- \frac{1}{2}\chi(r)\frac{b+\rho\cos(\theta-\eta)}{r}\partial_2\widetilde{\lambda}\Big) + \frac{\pi\rho}{2}\sin(\eta).
\end{aligned} \tag{20}$$

Moreover we have

$$\begin{aligned}
\|\widetilde{H'}\|_{\mathcal{H}^1_{\delta+1}} \lesssim& |b| + \|\dot{u}\nabla u\|_{H^0_{\delta+2}} + (1+|\alpha|+\|\widetilde{\lambda}\|_{H^2_\delta})\|\widetilde{\tau}\|_{H^1_{\delta+1}}\\
&+ |\rho| + (\|\widetilde{H}\|_{\mathcal{H}^1_{\delta+1}} + |b| + |\rho|)\|\widetilde{\lambda}\|_{H^2_\delta} + |\alpha|\|\widetilde{H}\|_{\mathcal{H}^1_{\delta+1}}.
\end{aligned}$$

To prove this proposition, we write $H' = H^{(1)} + H^{(2)} + H^{(3)}$ with

$$\begin{aligned}\partial_i H_{ij}^{(1)} =& -\dot{u}\partial_j u + \frac{1}{2}\partial_j \widetilde{\tau} - \frac{1}{2}\widetilde{\tau}\partial_j\lambda - \widetilde{H}_{ij}\partial_i\lambda \\ &+ \frac{\rho\chi'(r)}{4r}e_j - \partial_i\widetilde{\lambda}(H_b + H_{\rho,\eta})_{ij} - \frac{1}{2}\chi(r)\frac{b+\rho\cos(\theta-\eta)}{r}\partial_j\widetilde{\lambda},\end{aligned} \tag{21}$$

$$\partial_i H_{ij}^{(2)} = \frac{1}{2}\partial_j\left(\frac{b\chi(r)}{r}\right) + (H_b)_{ij}\partial_i(\alpha\chi(r)\ln(r)) + \frac{1}{2}\frac{b\chi(r)}{r}\partial_j(\alpha\chi(r)\ln(r)), \tag{22}$$

$$\begin{aligned}\partial_i H_{ij}^{(3)} =& \frac{1}{2}\partial_j\left(\frac{\rho\cos(\theta-\eta)\chi(r)}{r}\right) - \frac{\rho\chi'(r)}{4r}e_j + (H_{\rho,\eta})_{ij}\partial_i\left(\alpha\chi(r)\ln(r)\right) \\ &+ \frac{1}{2}\frac{\rho\chi(r)\cos(\theta-\eta)}{r}\partial_j(\alpha\chi(r)\ln(r)),\end{aligned} \tag{23}$$

where $e_1 = \cos(\eta)$ and $e_2 = \sin(\eta)$. The following three propositions allow us to solve (21), (22) and (23).

Proposition 3.2. *There exists a unique solution of* (21) *of the form*

$$H^{(1)} = \frac{m\chi(r)}{r}\begin{pmatrix}\cos(\theta+\phi) & \sin(\theta+\phi) \\ \sin(\theta+\phi) & -\cos(\theta+\phi)\end{pmatrix} + \widetilde{H}^{(1)},$$

with $\widetilde{H}^{(1)} \in H^1_{\delta+1}$ *and* $m\cos(\phi)$ *and* $m\sin(\phi)$ *are defined by* (19) *and* (20). *Moreover,* $\widetilde{H}^{(1)}$ *satisfies the estimate*

$$\begin{aligned}\|\widetilde{H'}\|_{\mathcal{H}^1_{\delta+1}} \lesssim& \|\dot{u}\nabla u\|_{H^0_{\delta+2}} + (1+|\alpha| + \|\widetilde{\lambda}\|_{H^2_\delta})\|\widetilde{\tau}\|_{H^1_{\delta+1}} \\ &+ |\rho| + (\|\widetilde{H}\|_{\mathcal{H}^1_{\delta+1}} + |b| + |\rho|)\|\widetilde{\lambda}\|_{H^2_\delta} + |\alpha|\|\widetilde{H}\|_{\mathcal{H}^1_{\delta+1}}.\end{aligned} \tag{24}$$

Proposition 3.3. *There exists an unique* $\widetilde{H}^{(2)} \in \mathcal{H}^1_{\delta+1}$ *such that* $H^{(2)} = H_b + \widetilde{H}^{(2)}$ *satisfies* (22). *Moreover we have the estimate*

$$\left\|\widetilde{H}^{(2)}\right\|_{\mathcal{H}^1_{\delta+1}} \lesssim |b|.$$

Proposition 3.4. *There exists an unique* $\widetilde{H}^{(3)} \in \mathcal{H}^1_{\delta+1}$ *such that*

$$H^{(3)} = -\frac{\rho\chi(r)}{4r}\begin{pmatrix}\cos(3\theta-\eta) & \sin(3\theta-\eta) \\ \sin(3\theta-\eta) & -\cos(3\theta-\eta)\end{pmatrix} + \widetilde{H}^{(3)},$$

satisfies (23). *Moreover, we have the estimate*

$$\left\|\widetilde{H}^{(3)}\right\|_{\mathcal{H}^1_{\delta+1}} \lesssim |\rho|.$$

Proposition 3.1 is a straightforward consequence of Propositions 3.2, 3.3 and 3.4. Thus, in the rest of this section, we prove Proposition 3.2, 3.3 and 3.4, respectively in section 3.1, 3.2 and 3.3.

3.1 Proof of Proposition 3.2

We need the following lemma.

Lemma 3.5. *Let $f_1, f_2 \in H^0_{\delta+2}$. The equation*

$$\partial_i K_{ij} = f_j,$$

with K a symmetric traceless tensor, has a unique solution of the form

$$K = \frac{m\chi(r)}{r}\begin{pmatrix} \cos(\theta+\phi) & \sin(\theta+\phi) \\ \sin(\theta+\phi) & -\cos(\theta+\phi) \end{pmatrix} + \widetilde{K},$$

with

$$m(\cos(\phi), \sin(\phi)) = \left(\int f_1, \int f_2\right)$$

and $\widetilde{K} \in \mathcal{H}^1_{\delta+1}$ with

$$\|\widetilde{K}\|_{H^1_{\delta+1}} \lesssim \|f_1\|_{H^0_{\delta+2}} + \|f_2\|_{H^0_{\delta+2}}.$$

We postpone the proof of Lemma 3.5 to the end of the section, and use it to prove Proposition 3.2.

Proof of Proposition 3.2. We apply Lemma 3.5 with

$$\begin{aligned} f_j = & -\dot{u}\partial_j u + \frac{1}{2}\partial_j\widetilde{\tau} - \frac{1}{2}\widetilde{\tau}\partial_j\lambda - \widetilde{H}_{ij}\partial_i\lambda + \frac{\rho\chi'(r)}{4r}e_j \\ & -\partial_i\widetilde{\lambda}(H_b + H_{\rho,\eta})_{ij} - \frac{1}{2}\chi(r)\frac{b+\rho\cos(\theta-\eta)}{r}\partial_j\widetilde{\lambda}, \quad j = 1,2. \end{aligned} \tag{25}$$

We first check that f_j belongs to $H^0_{\delta+2}$. Since $\widetilde{\tau} \in H^1_{\delta+1}$, we have $\partial_j\widetilde{\tau} \in H^0_{\delta+2}$ with

$$\|\partial_j\widetilde{\tau}\|_{H^0_{\delta+2}} \lesssim \|\widetilde{\tau}\|_{H^1_{\delta+1}}. \tag{26}$$

Moreover, thanks to Lemma 2.5, we have $\frac{\chi(r)}{r}\widetilde{\tau} \in H^1_{\delta+2}$. Since $\widetilde{\lambda} \in H^2_\delta$, we have $\partial_j\widetilde{\lambda} \in H^1_{\delta+1}$ and therefore, thanks to Proposition 2.4, $\widetilde{\tau}\partial_j\widetilde{\lambda} \in H^0_{\delta+2}$. Consequently we have the estimate

$$\|\widetilde{\tau}\partial_j\lambda\|_{H^0_{\delta+2}} \lesssim (|\alpha| + \|\widetilde{\lambda}\|_{H^2_\delta})\|\widetilde{\tau}\|_{H^1_{\delta+1}}. \tag{27}$$

In the same way, we have the estimates

$$\|\widetilde{H}_{ij}\partial_i\lambda\|_{H^0_{\delta+2}} \lesssim (|\alpha| + \|\widetilde{\lambda}\|_{H^2_\delta})\|\widetilde{H}\|_{\mathcal{H}^1_{\delta+1}}, \tag{28}$$

$$\left\|\partial_i\widetilde{\lambda}(H_b + H_{\rho,\eta})_{ij} - \frac{1}{2}\chi(r)\frac{b+\rho\cos(\theta-\eta)}{r}\partial_j\widetilde{\lambda}\right\|_{H^0_{\delta+2}} \lesssim (|b| + |\rho|)\|\widetilde{\lambda}\|_{H^2_\delta}. \tag{29}$$

Since χ' is compactly supported, we have

$$\left\|\frac{\rho\chi'(r)}{4r}e_j\right\|_{H^0_{\delta+2}} \lesssim |\rho|. \tag{30}$$

(26), (27), (28), (29) and (30) yield

$$\begin{aligned} \|f_1\|_{H^0_{\delta+2}} + \|f_2\|_{H^0_{\delta+2}} \lesssim & \|\dot{u}\nabla u\|_{H^0_{\delta+2}} + (1 + |\alpha| + \|\widetilde{\lambda}\|_{H^2_\delta})\|\widetilde{\tau}\|_{H^1_{\delta+1}} \\ & + |\rho| + (\|\widetilde{H}\|_{\mathcal{H}^1_{\delta+1}} + |b| + |\rho|)\|\widetilde{\lambda}\|_{H^2_\delta} + |\alpha|\|\widetilde{H}\|_{\mathcal{H}^1_{\delta+1}}. \end{aligned} \tag{31}$$

Therefore, Lemma 3.5 implies that we have a unique solution of (21) of the form

$$H^{(1)} = \frac{m\chi(r)}{r}\begin{pmatrix} \cos(\theta+\phi) & \sin(\theta+\phi) \\ \sin(\theta+\phi) & -\cos(\theta+\phi) \end{pmatrix} + \widetilde{H}^{(1)},$$

with $\widetilde{H}^{(1)} \in \mathcal{H}^1_{\delta+1}$. Together with (31), it yields the estimate (24). Then, in view of the definition (25) of f_1 and f_2, the computations

$$\int \frac{\rho\chi'(r)}{4r}\cos(\eta) r dr d\theta = \frac{\pi\rho}{2}\cos(\eta),$$
$$\int \frac{\rho\chi'(r)}{4r}\sin(\eta) r dr d\theta = \frac{\pi\rho}{2}\sin(\eta),$$

and the fact that

$$\int \partial_j \tilde{\tau} = 0,$$

yield the identities (19) and (20). □

Proof of lemma 3.5. We look for solutions of the form

$$K_{ij} = \partial_i Y_j + \partial_j Y_i - \delta_{ij}\partial^k Y_k. \tag{32}$$

Y then satisfies the equations

$$\Delta Y_1 = f_1,$$
$$\Delta Y_2 = f_2.$$

We can apply Corollary 2.7 which says that

$$Y_j = \left(\int f_j\right)\chi(r) ln(r) + \widetilde{Y}_j$$

where $\widetilde{Y}_j \in H^2_\delta$ satisfies

$$\|\widetilde{Y}_j\|_{H^2_\delta} \lesssim \|f_j\|_{H^0_{\delta+2}}.$$

We have then

$$K_{11} = \partial_1 Y_1 - \partial_2 Y_2 = \chi(r)\frac{x_1\int f_1 - x_2\int f_2}{r^2} + \widetilde{K}_{11},$$
$$K_{12} = \partial_1 Y_2 + \partial_2 Y_1 = \chi(r)\frac{x_1\int f_2 + x_2\int f_1}{r^2} + \widetilde{K}_{12},$$

where $\widetilde{K}_{11}, \widetilde{K}_{12} \in H^1_{\delta+1}$ satisfy

$$\|\widetilde{K}\|_{H^1_{\delta+1}} \lesssim \|f_1\|_{H^0_{\delta+2}} + \|f_2\|_{H^0_{\delta+2}}.$$

Let

$$m(\cos\phi, \sin\phi) = \left(\int f_1, \int f_2\right).$$

We obtain

$$\frac{x_1\int f_1 - x_2\int f_2}{r^2} = \frac{m}{r}(\cos\phi\cos\theta - \sin\phi\sin\theta) = \frac{m}{r}\cos(\theta+\phi),$$
$$\frac{x_1\int f_2 + x_2\int f_1}{r^2} = \frac{m}{r}(\sin\phi\cos\theta + \cos\phi\sin\theta) = \frac{m}{r}\sin(\theta+\phi).$$

This yields

$$K_{11} = \chi(r)\frac{m\cos(\theta+\phi)}{r} + \widetilde{K}_{11},$$
$$K_{12} = \chi(r)\frac{m\sin(\theta+\phi)}{r} + \widetilde{K}_{12}.$$

Since K is symmetric and traceless, we have $K_{22} = -K_{11}$ and $K_{12} = K_{21}$, which concludes the proof of Lemma 3.5. □

3.2 Proof of Proposition 3.3

We calculate the right-hand side of (22) for $j = 1$

$$\begin{aligned}
&\frac{1}{2}\partial_1\left(\frac{b\chi(r)}{r}\right) + (H_b)_{i1}\partial_i(\alpha\chi(r)\ln(r)) + \frac{1}{2}\frac{b\chi(r)}{r}\partial_1(\alpha\chi(r)\ln(r))\\
=&\frac{b}{2}\cos(\theta)\left(-\frac{\chi(r)}{r^2} + \frac{\chi'(r)}{r}\right) - \alpha\frac{b\chi(r)}{2r}\left(\chi'(r)\ln(r) + \frac{\chi(r)}{r}\right)(\cos(\theta)\cos(2\theta) + \sin(\theta)\sin(2\theta))\\
&+\alpha\frac{b\chi(r)}{2r}\left(\chi'(r)\ln(r) + \frac{\chi(r)}{r}\right)\cos(\theta)\\
=&\frac{b}{2}\cos(\theta)\left(-\frac{\chi(r)}{r^2} + \frac{\chi'(r)}{r}\right).
\end{aligned}$$

We have similarly for $j = 2$,

$$\begin{aligned}
&\frac{1}{2}\partial_2\left(\frac{b\chi(r)}{r}\right) + (H_b)_{i2}\partial_i(\alpha\chi(r)\ln(r)) + \frac{1}{2}\frac{b\chi(r)}{r}\partial_2(\alpha\chi(r)\ln(r))\\
=&\frac{b}{2}\sin(\theta)\left(-\frac{\chi(r)}{r^2} + \frac{\chi'(r)}{r}\right).
\end{aligned}$$

We calculate then,

$$\begin{aligned}
\partial_i(H_b)_{i1} =& -\frac{2b\chi(r)}{2r^2}(\sin(\theta)\sin(2\theta) + \cos(\theta)\cos(2\theta))\\
&-\frac{b}{2}\left(-\frac{\chi(r)}{r^2} + \frac{\chi'(r)}{r^2}\right)(\cos(\theta)\cos(2\theta) + \sin(\theta)\sin(2\theta))\\
=& -\frac{b\chi(r)}{2r^2}\cos(\theta) - \frac{b\chi'(r)}{2r}\cos(\theta).
\end{aligned}$$

In the same way

$$\partial_i(H_b)_{i2} = -\frac{b\chi(r)}{2r^2}\sin(\theta) - \frac{b\chi'(r)}{2r}\sin(\theta).$$

Therefore $H_b + \widetilde{H}^{(2)}$ satisfies (22) if and only if

$$\partial_i\widetilde{H}^{(2)}_{ij} = f_j, \tag{33}$$

with

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \frac{b\chi'(r)}{r}\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}.$$

We have $f_1, f_2 \in H^0_{\delta+2}$ with

$$\|f_1\|_{H^0_{\delta+2}} + \|f_2\|_{H^0_{\delta+2}} \lesssim |b|,$$

and

$$\int f_1 = \int f_2 = 0.$$

Therefore Lemma 3.5 implies that there exists a unique solution $\widetilde{H}^{(2)} \in H^1_{\delta+1}$ of (33). Furthermore it satisfies the estimate

$$\left\|\widetilde{H}^{(2)}\right\|_{\mathcal{H}^1_{\delta+1}} \lesssim |b|,$$

which concludes the proof of Proposition 3.3.

3.3 Proof of Proposition 3.4

We calculate each term of the right-hand side of (23) for $j = 1$

$$
\begin{aligned}
&\frac{1}{2}\partial_1\left(\frac{\rho\cos(\theta-\eta)\chi(r)}{r}\right)-\frac{\rho\chi'(r)}{4r}\cos(\eta)\\
&=\frac{\rho\chi(r)}{2r^2}\left(\sin(\theta)\sin(\theta-\eta)-\cos(\theta)\cos(\theta-\eta)\right)+\frac{\rho\chi'(r)}{2r}\cos(\theta)\cos(\theta-\eta)-\frac{\rho\chi'(r)}{4r}\cos(\eta)\\
&=-\frac{\rho\chi(r)}{2r^2}\cos(2\theta-\eta)+\frac{\rho\chi'(r)}{2r}\frac{\cos(\eta)+\cos(2\theta-\eta)}{2}-\frac{\rho\chi'(r)}{4r}\cos(\eta)\\
&=-\frac{\rho\chi(r)}{2r^2}\cos(2\theta-\eta)+\frac{\rho\chi'(r)}{4r}\cos(2\theta-\eta),
\end{aligned}
$$

$$
\begin{aligned}
&(H_{\rho,\eta})_{i1}\partial_i(\alpha\chi(r)\ln(r))+\frac{1}{2}\frac{\rho\chi(r)\cos(\theta-\eta)}{r}\partial_1(\alpha\chi(r)\ln(r))\\
=&-\frac{\alpha\rho\chi(r)}{4r}\left(\partial_r(\chi(r)\ln(r))\right)\left(\cos(\theta)(\cos(\theta+\eta)+\cos(3\theta-\eta))+\sin(\theta)(\sin(\theta+\eta)+\sin(3\theta-\eta))\right)\\
&+\frac{\alpha\rho\chi(r)}{2r}\left(\partial_r(\chi(r)\ln(r))\right)\cos(\theta)\cos(\theta-\eta)\\
=&-\frac{\alpha\rho\chi(r)}{4r}\left(\partial_r(\chi(r)\ln(r))\right)\left(\cos(\eta)+\cos(2\theta-\eta)\right)+\frac{\alpha\rho\chi(r)}{2r}\left(\partial_r(\chi(r)\ln(r))\right)\frac{\cos(\eta)+\cos(2\theta-\eta)}{2}\\
=&0.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
&\frac{1}{2}\partial_1\left(\frac{\rho\cos(\theta-\eta)\chi(r)}{r}\right)-\frac{\rho\chi'(r)}{4r}\cos(\eta)+(H_{\rho,\eta})_{i1}\partial_i(\alpha\chi(r)\ln(r))\\
&+\frac{1}{2}\frac{\rho\chi(r)\cos(\theta-\eta)}{r}\partial_1(\alpha\chi(r)\ln(r))\\
=&-\frac{\rho\chi(r)}{2r^2}\cos(2\theta-\eta)+\frac{\rho\chi'(r)}{4r}\cos(2\theta-\eta).
\end{aligned}
$$

In the same way, for $j = 2$ we have

$$
\begin{aligned}
&\frac{1}{2}\partial_2\left(\frac{\rho\cos(\theta-\eta)\chi(r)}{r}\right)-\frac{\rho\chi'(r)}{4r}\sin(\eta)+(H_{\rho,\eta})_{i2}\partial_i(\alpha\chi(r)\ln(r))\\
&+\frac{1}{2}\frac{\rho\chi(r)\cos(\theta-\eta)}{r}\partial_2(\alpha\chi(r)\ln(r))\\
=&-\frac{\rho\chi(r)}{2r^2}\sin(2\theta-\eta)+\frac{\rho\chi'(r)}{4r}\sin(2\theta-\eta).
\end{aligned}
$$

We calculate now

$$
\begin{aligned}
&\partial_1\left(-\frac{\rho\chi(r)\cos(3\theta-\eta)}{4r}\right)+\partial_2\left(-\frac{\rho\chi(r)\sin(3\theta-\eta)}{4r}\right)\\
=&-\frac{\rho}{4}(\cos(\theta)\cos(3\theta-\eta)+\sin(\theta)\sin(3\theta-\eta))\left(-\frac{\chi(r)}{r^2}+\frac{\chi'(r)}{r}\right)\\
&-3\frac{\rho}{4}\left(\sin(\theta)\sin(3\theta-\eta)+\cos(\theta)\cos(3\theta-\eta)\right)\frac{\chi(r)}{r^2}\\
=&-\frac{\rho\chi(r)}{2r^2}\cos(2\theta-\eta)-\frac{\rho\chi'(r)}{4r}\cos(2\theta-\eta).
\end{aligned}
$$

In the same way we have

$$\begin{aligned} &-\partial_2\left(-\frac{\rho\chi(r)\cos(3\theta-\eta)}{4r}\right)+\partial_1\left(-\frac{\rho\chi(r)\sin(3\theta-\eta)}{4r}\right) \\ &=-\frac{\rho\chi(r)}{2r^2}\sin(2\theta-\eta)-\frac{\rho\chi'(r)}{4r}\sin(2\theta-\eta). \end{aligned}$$

Therefore

$$H^{(3)}=-\frac{\rho\chi(r)}{4r}\begin{pmatrix}\cos(3\theta-\eta) & \sin(3\theta-\eta)\\ \sin(3\theta-\eta) & -\cos(3\theta-\eta)\end{pmatrix}+\widetilde{H}^{(3)}$$

satisfies (23) if and only if

$$\partial_i\widetilde{H}^{(3)}_{ij}=f_j, \tag{34}$$

with

$$\begin{pmatrix} f_1\\ f_2\end{pmatrix}=\frac{\rho\chi'(r)}{2r}\begin{pmatrix}\cos(2\theta-\eta)\\ \sin(2\theta-\eta)\end{pmatrix}.$$

We have $f_1,f_2\in H^0_{\delta+2}$ with

$$\|f_1\|_{H^0_{\delta+2}}+\|f_2\|_{H^0_{\delta+2}}\lesssim|\rho|,$$

and

$$\int f_1=\int f_2=0.$$

Therefore Lemma 3.5 implies that there exists a unique solution $\widetilde{H}^{(3)}\in H^1_{\delta+1}$ of (34). Furthermore, it satisfies the estimate

$$\left\|\widetilde{H}^{(3)}\right\|_{\mathcal{H}^1_{\delta+1}}\lesssim|\rho|,$$

which concludes the proof of Proposition 3.4.

4 The choice of ρ,η and the Lichnerowicz equation

The goal of this section is to solve equation (15).

4.1 The choice of ρ,η

We assume $\dot u\nabla u\in H^2_\delta$. Let $\alpha,b\in\mathbb{R}$, $\widetilde{\lambda}\in H^2_\delta$ and $\widetilde{H}\in\mathcal{H}^1_{\delta+1}$. We consider the map

$$\begin{aligned} G:\mathbb{R}\times[0,2\pi[&\to\mathbb{R}\times[0,2\pi[\\ (\rho,\eta)&\mapsto(-4m,\phi), \end{aligned}$$

with (m,ϕ) given by

$$\begin{aligned} m\cos(\phi)=&\int\left(-\dot u\partial_1 u-\frac12\widetilde{\tau}\partial_1\lambda-\widetilde{H}_{i1}\partial_i\lambda-\partial_i\widetilde{\lambda}(H_b+H_{\rho,\eta})_{i1}-\frac12\chi(r)\frac{b+\rho\cos(\theta-\eta)}{r}\partial_1\widetilde{\lambda}\right)\\ &+\frac{\pi\rho}{2}\cos(\eta),\\ m\sin(\phi)=&\int\left(-\dot u\partial_2 u-\frac12\widetilde{\tau}\partial_2\lambda-\widetilde{H}_{i2}\partial_i\lambda-\partial_i\widetilde{\lambda}(H_b+H_{\rho,\eta})_{i2}-\frac12\chi(r)\frac{b+\rho\cos(\theta-\eta)}{r}\partial_2\widetilde{\lambda}\right)\\ &+\frac{\pi\rho}{2}\sin(\eta). \end{aligned}$$

We want our solution H' of equation (14) to have the same form as H, in order to find a fixed point $H=H'$ and $\lambda=\lambda'$. Therefore, we need to show that G has a fixed point. This is done in the following lemma.

Lemma 4.1. *If $\|\widetilde{\lambda}\|_{H^2_\delta} \lesssim \varepsilon$, then for $\varepsilon > 0$ small enough, G admits an unique fixed point $(\rho,\eta) \in \mathbb{R}\times[0,2\pi[$. Moreover, if we assume*

$$\|\dot{u}\nabla u\|_{H^0_{\delta+2}} + \|\widetilde{H}\|_{\mathcal{H}^1_{\delta+1}} + \|\widetilde{\tau}\|_{H^1_{\delta+1}} + \|\widetilde{\lambda}\|_{H^2_\delta} + |b| + |\alpha| \lesssim \varepsilon, \tag{35}$$

then we have

$$\rho\cos(\eta) = \frac{4}{1+2\pi}\int \dot{u}\partial_1 u + O(\varepsilon^2),$$
$$\rho\sin(\eta) = \frac{4}{1+2\pi}\int \dot{u}\partial_2 u + O(\varepsilon^2).$$

Proof. We note

$$a_j = \int\left(-\dot{u}\partial_j u - \frac{1}{2}\widetilde{\tau}\partial_j\lambda - \widetilde{H}_{ij}\partial_i\lambda - \partial_i\widetilde{\lambda}(H_b)_{ij} - \frac{1}{2}\chi(r)\frac{b}{r}\partial_j\widetilde{\lambda}\right).$$

The conditions $\rho = -4m$ and $\eta = \phi$ are satisfied if and only if

$$\rho\cos(\eta) = -4\left(a_1 + \int\left(-\partial_i\widetilde{\lambda}(H_{\rho,\eta})_{i1} - \frac{1}{2}\chi(r)\frac{\rho\cos(\theta-\eta)}{r}\partial_1\widetilde{\lambda}\right) + \frac{\pi\rho}{2}\cos(\eta)\right), \tag{36}$$
$$\rho\sin(\eta) = -4\left(a_2 + \int\left(-\partial_i\widetilde{\lambda}(H_{\rho,\eta})_{i2} - \frac{1}{2}\chi(r)\frac{\rho\cos(\theta-\eta)}{r}\partial_2\widetilde{\lambda}\right) + \frac{\pi\rho}{2}\sin(\eta)\right). \tag{37}$$

Since we assume $\|\widetilde{\lambda}\|_{H^2_\delta} \lesssim \varepsilon$, we can write this system under the form

$$\begin{pmatrix} 1+O(\varepsilon) & O(\varepsilon) \\ O(\varepsilon) & 1+O(\varepsilon)\end{pmatrix}\begin{pmatrix}\rho\cos(\eta)\\ \rho\sin(\eta)\end{pmatrix} = -\frac{4}{1+2\pi}\begin{pmatrix} a_1 \\ a_2\end{pmatrix}.$$

For $\varepsilon > 0$ small enough, it is invertible, so G has a fixed point, and we obtain, under the hypothesis (35)

$$\rho\cos(\eta) = \frac{4}{1+2\pi}\int \dot{u}\partial_1 u + O(\varepsilon^2),$$
$$\rho\sin(\eta) = \frac{4}{1+2\pi}\int \dot{u}\partial_2 u + O(\varepsilon^2),$$

which concludes the proof of Lemma 4.1. ☐

4.2 The Lichnerowicz equation

Proposition 4.2. *Let $\dot{u}^2, |\nabla u|^2 \in H^0_{\delta+2}$. Let $\widetilde{\tau} \in H^1_{\delta+1}$, $\widetilde{H} \in \mathcal{H}^1_{\delta+1}$, $b,\rho,\eta \in \mathbb{R}$ and*

$$H = H_b + H_{\rho,\eta} + \widetilde{H},$$
$$\tau = \frac{b\chi(r)}{r} + \frac{\rho\chi(r)}{r}\cos(\theta-\eta) + \widetilde{\tau}.$$

There exists a unique λ' of the form

$$\lambda' = -\alpha'\chi(r)\ln(r) + \widetilde{\lambda}',$$

with $\widetilde{\lambda}' \in H^2_\delta$, solution of

$$\Delta\lambda' = -\frac{1}{2}\dot{u}^2 - \frac{1}{2}|\nabla u|^2 - \frac{1}{2}|H|^2 + \frac{\tau^2}{4}. \tag{38}$$

Moreover, if

$$\|\dot{u}^2\|_{H^0_{\delta+2}} + \||\nabla u|^2\|_{H^0_{\delta+2}} + \|\widetilde{H}\|_{\mathcal{H}^1_{\delta+1}} + \|\widetilde{\tau}\|_{H^1_{\delta+1}} + |b| + |\rho| \lesssim \varepsilon, \tag{39}$$

we have $\|\widetilde{\lambda}'\|_{H^2_\delta} \lesssim \varepsilon$ *and*

$$\alpha' = \frac{1}{2}\int \left(\dot{u}^2 + |\nabla u|^2\right) + O(\varepsilon^2).$$

Proof. We want to apply Corollary 2.7. We have to check that the right-hand side of (38) is in $H^0_{\delta+2}$. We write

$$\begin{aligned} |H|^2 &= |H_b + H_{\rho,\eta}|^2 + f_1, \\ \tau^2 &= \left(\frac{(b+\rho\cos(\theta-\eta))\chi(r)}{r}\right)^2 + f_2. \end{aligned} \tag{40}$$

We first estimate f_1 and f_2. Since $\widetilde{\tau} \in H^1_{\delta+1}$, $\widetilde{H} \in \mathcal{H}^1_{\delta+1}$, we have thanks to Proposition 2.4 that $\widetilde{\tau}^2, |\widetilde{H}|^2 \in H^0_{\delta+2}$, and thanks to Lemma 2.5,

$$\widetilde{\tau}\left(\frac{(b+\rho\cos(\theta-\eta))\chi(r)}{r}\right) \in H^0_{\delta+2}, \quad \widetilde{H}^{ij}(H_b + H_{\rho,\eta})_{ij} \in H^0_{\delta+2}.$$

Therefore, we have $f_1, f_2 \in H^0_{\delta+2}$ with

$$\|f_1\|_{H^0_{\delta+2}} + \|f_2\|_{H^0_{\delta+2}} \lesssim \|\widetilde{H}\|^2_{\mathcal{H}^1_{\delta+1}} + \|\widetilde{\tau}\|^2_{H^1_{\delta+1}} + b^2 + \rho^2. \tag{41}$$

We now compute $|H_b + H_{\rho,\eta}|^2$ and $\left(\frac{(b+\rho\cos(\theta-\eta))\chi(r)}{r}\right)^2$. We have

$$\begin{aligned} |H_b + H_{\rho,\eta}|^2 =& \frac{\chi(r)^2}{r^2}\Bigg(\frac{b^2}{2} + \frac{\rho^2}{4} + \frac{b\rho}{2}\left(\cos(2\theta)\cos(\theta+\eta) + \sin(2\theta)\sin(\theta+\eta)\right) \\ &+ \frac{b\rho}{2}\left(\cos(2\theta)\cos(3\theta-\eta) + \sin(2\theta)\sin(3\theta-\eta)\right) \\ &+ \frac{\rho^2}{4}\left(\cos(\theta+\eta)\cos(3\theta-\eta) + \sin(\theta+\eta)\sin(3\theta-\eta)\right)\Bigg) \\ =& \frac{\chi(r)^2}{r^2}\left(\frac{b^2}{2} + \frac{\rho^2}{4} + b\rho\cos(\theta-\eta) + \frac{\rho^2}{4}\cos(2\theta-\eta)\right), \end{aligned}$$

$$\begin{aligned} \left(\frac{(b+\rho\cos(\theta-\eta))\chi(r)}{r}\right)^2 =& \frac{\chi(r)^2}{r^2}\left(b^2 + 2b\rho\cos(\theta-\eta) + \rho^2\cos^2(\theta-\eta)\right) \\ =& \frac{\chi(r)^2}{r^2}\left(b^2 + 2b\rho\cos(\theta-\eta) + \frac{\rho^2}{2} + \frac{\rho^2}{2}\cos(2\theta-2\eta)\right). \end{aligned}$$

Therefore we have

$$\frac{1}{2}|H_b + H_{\rho,\eta}|^2 = \frac{1}{4}\left(\frac{(b+\rho\cos(\theta-\eta))\chi(r)}{r}\right)^2. \tag{42}$$

(40), (41) and (42) imply that the right-hand side of (38) is in $H^0_{\delta+2}$ with

$$\left\|-\frac{1}{2}\dot{u}^2 - \frac{1}{2}|\nabla u|^2 - \frac{1}{2}|H|^2 + \frac{\tau^2}{4}\right\|_{H^0_{\delta+2}} \lesssim \|\dot{u}^2\|_{H^0_{\delta+2}} + \||\nabla u|^2\|_{H^0_{\delta+2}} + \|\widetilde{H}\|^2_{\mathcal{H}^1_{\delta+1}} + \|\widetilde{\tau}\|^2_{H^1_{\delta+1}} + b^2 + \rho^2,$$

so Corollary 2.7 gives a unique solution of (38) of the form

$$\lambda' = -\alpha' \chi(r) \ln(r) + \widetilde{\lambda}',$$

with

$$\alpha' = \int \left(\frac{1}{2}\dot{u}^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|H|^2 - \frac{\tau^2}{4} \right),$$

and $\widetilde{\lambda}' \in H^2_\delta$. If we assume (39), we obtain

$$\alpha' = \frac{1}{2}\int \left(\dot{u}^2 + |\nabla u|^2\right) + O\left(\|f_1\|_{H^0_{\delta+2}} + \|f_2\|_{H^0_{\delta+2}}\right) = \frac{1}{2}\int \left(\dot{u}^2 + |\nabla u|^2\right) + O(\varepsilon^2),$$

which, together with the estimate

$$\|\widetilde{\lambda}'\|_{H^2_\delta} \lesssim \|\dot{u}^2\|_{H^0_{\delta+2}} + \||\nabla u|^2\|_{H^0_{\delta+2}} + \|f_1\|_{H^0_{\delta+2}} + \|f_2\|_{H^0_{\delta+2}} \lesssim \varepsilon,$$

concludes the proof of Proposition 4.2. $\square$

5 Proof of Theorem 2.10

We now have constructed, for $\varepsilon > 0$ small enough, a map

$$F : \mathbb{R} \times H^2_\delta \times \mathcal{H}^1_{\delta+1} \to \mathbb{R} \times H^2_\delta \times \mathcal{H}^1_{\delta+1}$$

which maps $(\alpha, \widetilde{\lambda}, \widetilde{H})$ satisfying

$$|\alpha| + \|\widetilde{\lambda}\|_{H^2_\delta} + \|\widetilde{H}\|_{\mathcal{H}^1_{\delta+1}} \lesssim \varepsilon$$

to $(\alpha', \widetilde{\lambda}', \widetilde{H'})$ such that, if ρ, η, depending on $\dot{u}, \nabla u, \widetilde{\tau}, \alpha, \lambda, \widetilde{H}, b$, are given by Lemma 4.1, then

$$H' = H_b + H_{\rho,\eta} + \widetilde{H}'$$

is the solution of

$$\partial_i H'_{ij} + H_{ij}\partial_i \lambda = -\dot{u}\partial_j u + \frac{1}{2}\partial_j \tau - \frac{1}{2}\tau \partial_j \lambda,$$

given by Proposition 3.1, with

$$\begin{aligned} H =& H_b + H_{\rho,\eta} + \widetilde{H}, \\ \lambda =& -\alpha\chi(r)\ln(r) + \widetilde{\lambda}, \\ \tau =& \frac{\chi(r)}{r}(b + \rho\cos(\theta - \eta)) + \widetilde{\tau}, \end{aligned}$$

and

$$\lambda' = -\alpha' \chi(r) \ln(r) + \widetilde{\lambda}'$$

is the solution of

$$\Delta \lambda' = -\frac{1}{2}\dot{u}^2 - \frac{1}{2}|\nabla u|^2 - \frac{1}{2}|H|^2 + \frac{\tau^2}{4}$$

given by Proposition 4.2. Lemma 4.1 implies $|\rho| \lesssim \varepsilon$, and therefore Proposition 3.1 and Proposition 4.2 imply

$$|\alpha'| + \|\widetilde{\lambda}'\|_{H^2_\delta} + \|\widetilde{H'}\|_{\mathcal{H}^1_{\delta+1}} \lesssim \varepsilon.$$

In particular $\|F(0)\|_{\mathbb{R}\times H^2_\delta\times\mathcal{H}^1_{\delta+1}} \lesssim \varepsilon$, so there exists a constant $C \lesssim 1$ such that

$$\|F(0)\|_{\mathbb{R}\times H^2_\delta\times\mathcal{H}^1_{\delta+1}} = C\varepsilon.$$

Next, we show that F is a contracting map in $B_{\mathbb{R}\times H^2_\delta\times\mathcal{H}^1_{\delta+1}}(0,2C\varepsilon)$. We take, for $i=1,2$, $(\alpha_i,\widetilde{\lambda}_i,\widetilde{H}_i)\in\mathbb{R}\times H^2_\delta\times\mathcal{H}^1_{\delta+1}$ such that

$$|\alpha_i| + \|\widetilde{\lambda}_i\|_{H^2_\delta} + \|\widetilde{H}_i\|_{\mathcal{H}^1_{\delta+1}} \lesssim \varepsilon.$$

We note ρ_i, η_i the corresponding quantities given by Lemma 4.1. Thanks to formula (36) and (37) we get

$$\begin{aligned}
|\rho_1\cos(\eta_1)-\rho_2\cos(\eta_2)| \lesssim{}& \varepsilon\left(|\alpha_1-\alpha_2| + \|\widetilde{\lambda}_1-\widetilde{\lambda}_2\|_{H^2_\delta} + \|\widetilde{H}_1-\widetilde{H}_2\|_{\mathcal{H}^1_{\delta+1}}\right)\\
&+\varepsilon\left(|\rho_1\cos(\eta_1)-\rho_2\cos(\eta_2)| + |\rho_1\sin(\eta_1)-\rho_2\sin(\eta_2)|\right),\\
|\rho_1\sin(\eta_1)-\rho_2\sin(\eta_2)| \lesssim{}& \varepsilon\left(|\alpha_1-\alpha_2| + \|\widetilde{\lambda}_1-\widetilde{\lambda}_2\|_{H^2_\delta} + \|\widetilde{H}_1-\widetilde{H}_2\|_{\mathcal{H}^1_{\delta+1}}\right)\\
&+\varepsilon\left(|\rho_1\cos(\eta_1)-\rho_2\cos(\eta_2)| + |\rho_1\sin(\eta_1)-\rho_2\sin(\eta_2)|\right),
\end{aligned}$$

and so

$$\begin{aligned}
|\rho_1\cos(\eta_1)-\rho_2\cos(\eta_2)| \lesssim{}& \varepsilon\left(|\alpha_1-\alpha_2| + \|\widetilde{\lambda}_1-\widetilde{\lambda}_2\|_{H^2_\delta} + \|\widetilde{H}_1-\widetilde{H}_2\|_{\mathcal{H}^1_{\delta+1}}\right),\\
|\rho_1\sin(\eta_1)-\rho_2\sin(\eta_2)| \lesssim{}& \varepsilon\left(|\alpha_1-\alpha_2| + \|\widetilde{\lambda}_1-\widetilde{\lambda}_2\|_{H^2_\delta} + \|\widetilde{H}_1-\widetilde{H}_2\|_{\mathcal{H}^1_{\delta+1}}\right).
\end{aligned}$$

Thanks to Propositions 3.2, 3.3 and 3.4, we can write

$$H'_i = H_b - \frac{\rho_i\chi(r)}{4r}\begin{pmatrix}\cos(3\theta-\eta_i) & \sin(3\theta-\eta_i)\\ \sin(3\theta-\eta_i) & -\cos(3\theta-\eta_i)\end{pmatrix} + K_i,$$

where K_i satisfies

$$\begin{aligned}
\partial_l(K_i)_{lj} ={}& -\dot{u}\partial_j u + \frac{1}{2}\partial_j\widetilde{\tau} - \frac{1}{2}\widetilde{\tau}\partial_j\lambda_i - (\widetilde{H}_i)_{lj}\partial_l\lambda_i\\
&+(g_i)_j - \partial_l\widetilde{\lambda}_i(H_b+H_{\rho_i,\eta_i})_{lj} - \frac{1}{2}\chi(r)\frac{b+\rho_i\cos(\theta-\eta_i)}{r}\partial_j\widetilde{\lambda}_i,
\end{aligned}$$

with

$$\begin{pmatrix}(g_i)_1\\(g_i)_2\end{pmatrix} = \frac{\chi'(r)}{r}\begin{pmatrix} b\cos(\theta)+\frac{\rho_i}{2}\cos(2\theta-\eta_i)+\frac{\rho_i}{4}\cos(\eta_i)\\ b\sin(\theta)+\frac{\rho_i}{2}\sin(2\theta-\eta_i)+\frac{\rho_i}{4}\cos(\eta_i)\end{pmatrix}.$$

Thus, we can estimate

$$\begin{aligned}
\|\partial_l(K_1-K_2)_{lj}\|_{H^0_{\delta+2}} \lesssim{}& \varepsilon\left(|\alpha_1-\alpha_2| + \|\widetilde{\lambda}_1-\widetilde{\lambda}_2\|_{H^2_\delta} + \|\widetilde{H}_1-\widetilde{H}_2\|_{\mathcal{H}^1_{\delta+1}}\right)\\
&+|\rho_1\cos(\eta_1)-\rho_2\cos(\eta_2)| + |\rho_1\sin(\eta_1)-\rho_2\sin(\eta_2)|\\
\lesssim{}& \varepsilon\left(|\alpha_1-\alpha_2| + \|\widetilde{\lambda}_1-\widetilde{\lambda}_2\|_{H^2_\delta} + \|\widetilde{H}_1-\widetilde{H}_2\|_{\mathcal{H}^1_{\delta+1}}\right).
\end{aligned}$$

Therefore, we can apply Lemma 3.5 which yields

$$K_1-K_2 = \frac{m\chi(r)}{r}\begin{pmatrix}\cos(\theta+\phi) & \sin(\theta+\phi)\\ \sin(\theta+\phi) & -\cos(\theta+\phi)\end{pmatrix} + \widetilde{K},$$

with $\widetilde{K} \in \mathcal{H}^1_{\delta+1}$ which satisfies

$$\|\widetilde{K}\|_{\mathcal{H}^1_{\delta+1}} \lesssim \varepsilon \left(|\alpha_1 - \alpha_2| + \|\widetilde{\lambda}_1 - \widetilde{\lambda}_2\|_{H^2_\delta} + \|\widetilde{H}_1 - \widetilde{H}_2\|_{\mathcal{H}^1_{\delta+1}} \right).$$

By uniqueness of the decomposition given by Lemma 3.5, we obtain

$$\widetilde{K} = \widetilde{H}_1 - \widetilde{H}_2,$$

and we deduce

$$\|\widetilde{H}_1 - \widetilde{H}_2\|_{\mathcal{H}^1_{\delta+1}} \lesssim \varepsilon \left(|\alpha_1 - \alpha_2| + \|\widetilde{\lambda}_1 - \widetilde{\lambda}_2\|_{H^2_\delta} + \|\widetilde{H}_1 - \widetilde{H}_2\|_{\mathcal{H}^1_{\delta+1}} \right). \tag{43}$$

Last we estimate

$$\begin{aligned}\|\Delta\lambda'_1 - \Delta\lambda'_2\|_{H^2_\delta} \lesssim & \varepsilon \|\widetilde{H}_1 - \widetilde{H}_2\|_{\mathcal{H}^1_{\delta+1}} + \varepsilon \left(|\rho_1 \cos(\eta_1) - \rho_2 \cos(\eta_2)| + |\rho_1 \sin(\eta_1) - \rho_2 \sin(\eta_2)| \right) \\ \lesssim & \varepsilon \left(|\alpha_1 - \alpha_2| + \|\widetilde{\lambda}_1 - \widetilde{\lambda}_2\|_{H^2_\delta} + \|\widetilde{H}_1 - \widetilde{H}_2\|_{\mathcal{H}^1_{\delta+1}} \right).\end{aligned}$$

Thus, thanks to Corollary 2.7, we have

$$|\alpha'_1 - \alpha'_2| + \|\widetilde{\lambda'}_1 - \widetilde{\lambda'}_2\|_{H^2_\delta} \lesssim \varepsilon \left(|\alpha_1 - \alpha_2| + \|\widetilde{\lambda}_1 - \widetilde{\lambda}_2\|_{H^2_\delta} + \|\widetilde{H}_1 - \widetilde{H}_2\|_{\mathcal{H}^1_{\delta+1}} \right). \tag{44}$$

In view of (43) and (44), for $\varepsilon > 0$ small enough, F is a contracting map in

$$B_{\mathbb{R} \times H^2_\delta \times \mathcal{H}^1_{\delta+1}}(0, 2C\varepsilon).$$

Therefore, F has a unique fixed point $(\alpha, \widetilde{\lambda}, \widetilde{H})$. The estimates of Lemma 4.1 and Proposition 4.2 complete the proof of Theorem 2.10.

References

[1] Abhay Ashtekar, Jiří Bičák, and Bernd G. Schmidt. Asymptotic structure of symmetry-reduced general relativity. *Phys. Rev. D (3)*, 55(2):669–686, 1997.

[2] Murray Cantor. A necessary and sufficient condition for York data to specify an asymptotically flat spacetime. *J. Math. Phys.*, 20(8):1741–1744, 1979.

[3] Yvonne Choquet-Bruhat and Vincent Moncrief. Nonlinear stability of an expanding universe with the S^1 isometry group. In *Partial differential equations and mathematical physics (Tokyo, 2001)*, volume 52 of *Progr. Nonlinear Differential Equations Appl.*, pages 57–71. Birkhäuser Boston, Boston, MA, 2003.

[4] Yvonne Choquet-Bruhat. *General relativity and the Einstein equations.* Oxford Mathematical Monographs. Oxford University Press, Oxford, 2009.

[5] Demetrios Christodoulou and Sergiu Klainerman. *The global nonlinear stability of the Minkowski space*, volume 41 of *Princeton Mathematical Series.* Princeton University Press, Princeton, NJ, 1993.

[6] David Maxwell. A class of solutions of the vacuum Einstein constraint equations with freely specified mean curvature. *Math. Res. Lett.*, 16(4):627–645, 2009.

[7] Robert C. McOwen. The behavior of the Laplacian on weighted Sobolev spaces. *Comm. Pure Appl. Math.*, 32(6):783–795, 1979.

[8] Wei Ming Ni. On the elliptic equation $\Delta u + K(x)e^{2u} = 0$ and conformal metrics with prescribed Gaussian curvatures. *Invent. Math.*, 66(2):343–352, 1982.

[9] Michael E. Taylor. *Partial differential equations III. Nonlinear equations*, volume 117 of *Applied Mathematical Sciences*. Springer, New York, second edition, 2011.